

QUASIPOLARITY OF GENERALIZED MATRIX RINGS

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ABSTRACT. An element a of a ring R is called *quasipolar* provided that there exists an idempotent $p \in R$ such that $p \in \text{comm}^2(a)$, $a+p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. In this paper, we investigate quasipolarity of generalized matrix rings $K_s(R)$ for a commutative local ring R and $s \in R$. We show that if s is nilpotent, then $K_s(R)$ is quasipolar. We determine the conditions under which elements of $K_s(R)$ are quasipolar. It is shown that $K_s(R)$ is quasipolar if and only if $\text{tr}(A) \in J(R)$ or the equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R for every $A \in K_s(R)$ with $\det_s(A) \in J(R)$. Furthermore, we prove that $M_2(R)$ is quasipolar if and only if $M_2(R)$ is strongly clean for a commutative local ring R .

Keywords: Quasipolar ring, local ring, generalized matrix ring.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity unless otherwise stated. Following Koliha and Patricio [5], the *commutant* and *double commutant* of $a \in R$ are defined by $\text{comm}(a) = \{x \in R \mid xa = ax\}$, $\text{comm}^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a)\}$, respectively. If $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a)\}$ and $a \in R^{qnil}$, then a is said to be *quasinilpotent* [4]. An element $a \in R$ is called *quasipolar* provided that there exists an idempotent $p \in R$ such that $p \in \text{comm}^2(a)$, $a+p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. Some properties of quasipolar rings were studied in [2, 3, 5, 10].

Let R be a ring, and let $s \in R$ be central. Following Krylov [6], we use $K_s(R)$ to denote the set $\{[a_{ij}] \in M_2(R) \mid \text{each } a_{ij} \in R\}$ with the following operations:

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} &= \begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}, \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} &= \begin{bmatrix} aa' + sbc' & ab' + bd' \\ ca' + dc' & scb' + dd' \end{bmatrix}. \end{aligned}$$

The element s is called *the multiplier of $K_s(R)$* . The set $K_s(R)$ becomes a ring with these operations and can be viewed as a special kind of Morita context. A Morita context (A, B, M, N, ψ, ϕ) consists of two rings A and B , two bimodules ${}_A M_B$, ${}_B N_A$ and a pair of bimodule homomorphisms $\psi : M \otimes_B N \rightarrow A$ and $\phi : N \otimes_A M \rightarrow B$ which satisfy the following associativity: $\psi(m \otimes n)m' = m\phi(n \otimes m')$ and $\phi(n \otimes m)n' = n\psi(m \otimes n')$ for any $n, n' \in N$, $m, m' \in M$. These conditions insure that the set T of generalized matrices $\begin{bmatrix} a & m \\ n & b \end{bmatrix}$; $a \in A$, $b \in B$, $m \in M$, $n \in N$ will form a ring,

called the ring of the *Morita context*. A Morita context $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring over R* . It was observed by Krylov [6] that the generalized matrix rings over R are precisely these rings $K_s(R)$ with $s \in C(R)$. When $s = 1$, $K_1(R)$ is just the matrix ring $M_2(R)$, but $K_s(R)$ can be significantly different from $M_2(R)$. In fact, for a local ring R and $s \in C(R)$, $K_s(R) \cong K_1(R)$ if and only if $s \in U(R)$ (see [6, Lemma 3 and Corollary 2] and [8, Corollary 4.10]). Some properties of the ring $K_s(R)$ is studied comprehensively by Krylov and Tuganbaev in [7].

In this paper, we study the quasipolarity of the generalized matrix ring $K_s(R)$ over a commutative local ring R . It is shown that $K_s(R)$ is quasipolar if and only if $\text{tr}(A) \in J(R)$ or the equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R for every $A \in K_s(R)$ with $\det_s(A) \in J(R)$. This yields the main result of [2] for $s = 1$. Furthermore, for $s \in U(R)$, we show that $K_s(R)$ is quasipolar if and only if $K_s(R)$ is strongly clean. In particular, a criterion for the quasipolarity of $K_s(R[[x]])$ is obtained. We see that if s is a nilpotent element in R , then $K_s(R)$ is quasipolar.

In what follows, the ring of integers modulo n is denoted by \mathbb{Z}_n , and we write $M_n(R)$ (resp. $T_n(R)$) for the rings of all (resp., all upper triangular) $n \times n$ matrices over the ring R . We write $R[[x]]$, $U(R)$, $C(R)$, R^{nil} and $J(R)$ for the power series ring over a ring R , the set of all invertible elements, the set of all central elements, the set of all nilpotent elements and the Jacobson radical of R , respectively. For a prime integer p , $\mathbb{Z}_{(p)}$ will be the localization of \mathbb{Z} at p and the ring $\widehat{\mathbb{Z}}_p$ denotes p -adic integers.

2. GENERALIZED MATRIX RINGS

In this section we study the quasipolarity of the generalized matrix ring over a commutative local ring. The quasipolarity of $K_s(R)$ strictly depends on the values of s . Namely, $s \in U(R)$ if and only if $K_s(R) \cong M_2(R)$ for any local ring R with $s \in C(R)$. We prove that if s is a nilpotent element of R , then $K_s(R)$ is always quasipolar. We supply an example to show that $K_s(R)$ is quasipolar if and only if $s \in R^{nil}$ (see Example 2.22). And if $s \in U(R)$, then $K_s(\mathbb{Z}_{(p)})$ is not quasipolar.

Lemma 2.1. [9, Lemma 1] *Let R be a ring and let $s \in C(R)$. Then*

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix} \mapsto \begin{bmatrix} b & y \\ x & a \end{bmatrix} \text{ is an automorphism of } K_s(R).$$

If R is a commutative ring with $s \in R$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_s(R)$, we define $\det_s(A) = ad - sbc$ and $\text{tr}(A) = a + d$, and $rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$ for $r \in R$ (see [9]).

For elements a, b in a ring R , we use the notation $a \sim b$ to mean that a is similar to b , that is, $b = u^{-1}au$ for some $u \in U(R)$.

Lemma 2.2. [9, Lemma 14] *Let R be a commutative ring with $s \in R$ and let $A, B \in K_s(R)$. The following hold:*

- (1) $\det_s(AB) = \det_s(A)\det_s(B)$.
- (2) $A \in U(K_s(R))$ if and only if $\det_s(A) \in U(R)$. In this case, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \det_s(A)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(3) If $A \sim B$, then $\det_s(A) = \det_s(B)$ and $\text{tr}(A) = \text{tr}(B)$.

Note that $\det_s(I_2 + A) = 1 + \text{tr}(A) + \det_s(A)$ and $A^2 - \text{tr}(A)A + \det_s(A)I_2 = 0$ for any $A \in K_s(R)$.

We need Theorem 2.3 in the sequel and give a short proof for the sake of completeness.

Theorem 2.3. *Let R be a commutative local ring with $s \in R$ and let $A \in K_s(R)$ such that $\det_s(A) \in J(R)$. Then $\text{tr}(A) \in J(R)$ if and only if $A \in (K_s(R))^{qnil}$.*

Proof. Let $A \in K_s(R)$ such that $\det_s(A) \in J(R)$. Assume that $\text{tr}(A) \in J(R)$. Since $A^2 - \text{tr}(A)A + \det_s(A)I_2 = 0$, we have $A^2 = \text{tr}(A)A - \det_s(A)I_2 \in J(K_s(R))$. Let $X \in \text{comm}(A)$. Then $(I_2 - XA)(I_2 + XA) = I_2 - X^2A^2 \in U(R)$ and so $I_2 - XA \in U(K_s(R))$ because $A^2 \in J(K_s(R))$. Hence $A \in (K_s(R))^{qnil}$. Conversely, suppose that $A \in (K_s(R))^{qnil}$ and let $x \in R$. Since $A \in (K_s(R))^{qnil}$, $I_2 + xA \in U(K_s(R))$ and so $\det_s(I_2 + xA) = 1 + x\text{tr}(A) + x^2\det_s(A) \in U(R)$. As $\det_s(A) \in J(R)$, we have $1 + x\text{tr}(A) \in U(R)$. This gives $\text{tr}(A) \in J(R)$. \square

Lemma 2.4. *Let R be a ring and $s \in C(R)$. Then $A \in K_s(R)$ is quasipolar if and only if $PAP^{-1} \in K_s(R)$ is quasipolar for some $P \in U(K_s(R))$.*

Proof. It follows from [2, Lemma 2.3]. \square

For elements a, b in a ring R , we say that a is equivalent to b if there exist $u, v \in U(R)$ such that $b = uav$. Recall that an element $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is called a diagonal matrix of $K_s(R)$.

Lemma 2.5. [9, Lemma 3] *Let $E^2 = E \in K_s(R)$. If E is equivalent to a diagonal matrix in $K_s(R)$, then E is similar to a diagonal matrix in $K_s(R)$.*

Lemma 2.6. *Let R be a local ring with $s \in C(R)$ and let E be a non-trivial idempotent of $K_s(R)$. Then we have the following.*

(1) If $s \in U(R)$, then $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(2) If $s \in J(R)$, then either $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $E \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. Let $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in R$. Since $E^2 = E$, we have

$$a^2 + sbc = a, \quad scb + d^2 = d, \quad ab + bd = b, \quad ca + dc = c.$$

If $a, d \in J(R)$, then $scb, sbc, b, c \in J(R)$ and so $E \in J(K_s(R))$. Hence $E = 0$, a contradiction. Since R is local, we have $a \in U(R)$ or $d \in U(R)$. If $d \in U(R)$, then $\begin{bmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -d^{-1}c & d^{-1} \end{bmatrix} = \begin{bmatrix} a - sbd^{-1}c & 0 \\ 0 & 1 \end{bmatrix}$ and so E is equivalent to a diagonal matrix. If $a \in U(R)$, then we similarly show that E is equivalent to a diagonal matrix (see [9, Lemma 4]). According to Lemma 2.5, there exists $P \in U(K_s(R))$ such that $PEP^{-1} = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$, where $f, g \in R$ are idempotents. Since R is local and $E \neq 0$, we see that either $f = 1$ and $g = 0$ or $f = 0$ and $g = 1$. If $f = 1$ and $g = 0$, then $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let $f = 0$ and $g = 1$. If $s \in U(R)$, then $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & s^{-1} \\ s^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and so $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If $s \in J(R)$ and $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then it is easy to check that $s \in U(R)$, a contradiction. Hence, if $s \in J(R)$, then either $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $E \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. \square

Lemma 2.7. *Let R be a uniquely bleached local ring and $s \in C(R)$. Then $A \in K_s(R)$ is quasipolar if and only if either $A \in U(K_s(R))$ or $A \in (K_s(R))^{qnil}$ or $A \sim \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in R$.*

Proof. Suppose that A is quasipolar in $K_s(R)$. Then there exists $E^2 = E \in \text{comm}^2(A)$ such that $A + E = W \in U(K_s(R))$ and $EA \in (K_s(R))^{qnil}$. If $E = 0$ or $E = I_2$, then $A \in U(K_s(R))$ or $A \in (K_s(R))^{qnil}$, respectively. Hence, by Lemma 2.6, either $E \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $E \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Assume that

$E \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. That is, there exists $P \in U(K_s(R))$ such that $PEP^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. According to Lemma 2.4, $PAP^{-1} + PEP^{-1} = PWP^{-1}$ is a quasipolar decomposition in $K_s(R)$. Let $V = [v_{ij}] = PWP^{-1}$. Since $V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V$, we have $v_{12} = v_{21} = 0$ and so $A \sim \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in R$. Conversely, it suffices to show that $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is quasipolar in $K_s(R)$ by Lemma 2.4. Since R is a local ring, R is quasipolar by [10, Corollary 3.3]. Then there exist $e_1^2 = e_1 \in \text{comm}^2(a)$ and $e_2^2 = e_2 \in \text{comm}^2(b)$ such that $a + e_1 \in U(R)$, $b + e_2 \in U(R)$, $ae_1 \in R^{qnil}$ and $be_2 \in R^{qnil}$. Let $F = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$. It can be easily check that $F \in \text{comm}^2(B)$ and if $X \in \text{comm}(BF)$, then $I_2 + XBF \in U(K_s(R))$ and so $BF \in (K_s(R))^{qnil}$. This implies that $B + F = \begin{bmatrix} a + e_1 & 0 \\ 0 & b + e_2 \end{bmatrix}$ is a quasipolar decomposition. \square

Lemma 2.8. *Let R be a local ring with $s \in J(R) \cap C(R)$. Let $A \in K_s(R)$ such that $A \notin U(K_s(R))$ and $A \notin (K_s(R))^{qnil}$. Then A is similar to $\begin{bmatrix} u & 1 \\ v & w \end{bmatrix}$ or $\begin{bmatrix} w & 1 \\ v & u \end{bmatrix}$ where $u, v \in U(R)$ and $w \in J(R)$.*

Proof. It is similar to the proof of [9, Lemma 5]. \square

Lemma 2.9. *Let R be a commutative local ring with $s \in R$. Then every upper triangular matrix in $K_s(R)$ is quasipolar.*

Proof. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in K_s(R)$ where $a, b, c \in R$. We can assume that $\det_s(A) = ac \in J(R)$ and $\text{tr}(A) = a + c \in U(R)$ by Theorem 2.3. This gives $c - a \in U(R)$. Choose $P = \begin{bmatrix} 1 & b(c - a)^{-1} \\ 0 & 1 \end{bmatrix}$. By Lemma 2.2, $P \in U(K_s(R))$, and a direct calculation shows that $P^{-1}AP = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$. Then A is quasipolar from Lemma 2.4 and Lemma 2.7. \square

Theorem 2.10. *Let R be a commutative local ring with $s \in R$ and let $A \in K_s(R)$ such that neither $A \in U(K_s(R))$ nor $A \in (K_s(R))^{qnil}$. Then the following statements are equivalent:*

- (1) *A is quasipolar in $K_s(R)$.*
- (2) *The equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R .*

Proof. (1) \Rightarrow (2) Assume that $A \in K_s(R)$ is quasipolar. Since $A \notin U(K_s(R))$ and $A \notin (K_s(R))^{qnil}$, $A \sim B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in R$ by Lemma 2.7. According to Lemma 2.2, $\text{tr}(A) = \text{tr}(B)$ and $\det_s(A) = \det_s(B)$. This gives $x^2 - \text{tr}(A)x + \det_s(A) = x^2 - \text{tr}(B)x + \det_s(B)$. Since $a^2 - \text{tr}(B)a + \det_s(B) = 0$, the equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R .

(2) \Rightarrow (1) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in K_s(R)$ and suppose that the equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ has roots $a, b \in R$. Since $A \notin U(K_s(R))$ and $A \notin (K_s(R))^{qnil}$, $\det_s(A) = a_{11}a_{22} - sa_{12}a_{21} = ab \in J(R)$ and $\text{tr}(A) = a_{11} + a_{22} = a + b \in U(R)$ by Theorem 2.3. So one of a, b must be in $U(R)$ and the other must be in $J(R)$. Let $B = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix}$. Then $\text{tr}(B) = \text{tr}(A)$ and $\det_s(B) = \det_s(A)$, and A is quasipolar if and only if B is quasipolar by Lemma 2.1. Hence, without loss of generality we may assume that $a \in U(R)$, $b \in J(R)$ and $a_{11} \in U(R)$. Let $P = \begin{bmatrix} 1 & 0 \\ a_{21}(a - a_{22})^{-1} & 1 \end{bmatrix}$. By Lemma 2.2(2), $P \in U(K_s(R))$ and easy calculation shows that $P^{-1}AP$ is an upper triangular matrix in $K_s(R)$. Therefore A is quasipolar from Lemma 2.4 and Lemma 2.9. \square

Theorem 2.11. *Let R be a commutative local ring with $s \in R$. The following are equivalent.*

- (1) *$K_s(R)$ is quasipolar.*
- (2) *For every $A \in K_s(R)$ with $\det_s(A) \in J(R)$, one of the following holds:*
 - (i) *$\text{tr}(A) \in J(R)$,*
 - (ii) *The equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R .*

Proof. (1) \Rightarrow (2) Suppose that $A \in K_s(R)$ with $\det_s(A) \in J(R)$. By (1), there exists an idempotent $E \in K_s(R)$ such that $E \in \text{comm}^2(A)$ and $A+E \in U(K_s(R))$ and $EA \in (K_s(R))^{qnil}$. If $E = I_2$, then $A \in (K_s(R))^{qnil}$ and so $\text{tr}(A) \in J(R)$ by Theorem 2.3. So we can assume that $A \notin (K_s(R))^{qnil}$. Since $\det_s(A) \in J(R)$, $A \notin U(K_s(R))$ by Lemma 2.2(2). According to Theorem 2.10, the equation $x^2 - \text{tr}(A)x + \det_s(A) = 0$ is solvable in R .

(2) \Rightarrow (1) Let $A \in K_s(R)$. If $\det_s(A) \in U(R)$, then $A \in U(K_s(R))$ and so A is quasipolar. Let $\det_s(A) \in J(R)$. If $\text{tr}(A) \in J(R)$, then A is quasipolar by Theorem 2.3. Hence we assume that $\text{tr}(A) \in U(R)$. This gives $A \notin U(K_s(R))$ and $A \notin (K_s(R))^{qnil}$. By Theorem 2.10, A is quasipolar and so $K_s(R)$ is quasipolar. \square

Letting $s = 1$ in Theorem 2.11 yields the main result of [2].

Corollary 2.12. *Let R be a commutative local ring with $s \in U(R)$. The following are equivalent.*

- (1) $K_s(R)$ is quasipolar.
- (2) $K_s(R)$ is strongly clean.

Proof. (1) \Rightarrow (2) is obvious. (2) \Rightarrow (1) Let $A \in K_s(R)$. According to Theorem 2.11, we may assume that $\det_s(A) \in J(R)$ and $\text{tr}(A) \in U(R)$. By [9, Corollary 16], the equation $t^2 - t - w = 0$ is solvable in R for all $w \in J(R)$. Then the equation $t^2 - t - (\text{tr}(A))^{-2} \det_s(A) = 0$ is solvable in R . Hence the equation $t^2 - \text{tr}(A)t + \det_s(A) = 0$ is solvable in R . In view of Theorem 2.11, $K_s(R)$ is quasipolar. \square

By using Corollary 2.12, from [1, Theorem 2.4], for the ring $R = \widehat{\mathbb{Z}}_p$ and $s \in U(R)$, we have $K_s(R)$ is quasipolar if and only if $K_s(R)$ is strongly clean.

Corollary 2.13. *Let R be a commutative local ring. Then the following are equivalent.*

- (1) $K_1(R)$ is quasipolar.
- (2) $K_1(R)$ is strongly clean.
- (3) $K_s(R)$ is strongly clean for all $s \in R$.
- (4) $K_s(R)$ is strongly clean for all $s \in J(R)$.

Proof. Combine Corollary 2.12 with [9, Corollary 19]. \square

Corollary 2.13 shows that $M_2(R)$ is quasipolar if and only if $M_2(R)$ is strongly clean for a commutative local ring R .

Theorem 2.14. *Let R be a commutative local ring with $s \in J(R)$. Then the following are equivalent.*

- (1) $K_s(R)$ is a quasipolar ring.
- (2) For any $u, v \in U(R)$ and $w \in J(R)$, the equation $t^2 - (u + w)t + (uw - sv) = 0$ is solvable in R .
- (3) For any $u \in U(R)$ and $w \in J(R)$, the equation $t^2 - (1 + w)t + (w - su) = 0$ is solvable in R .
- (4) $\begin{bmatrix} 1 & 1 \\ u & w \end{bmatrix}$ is quasipolar in $K_s(R)$ for all $u \in U(R)$ and $w \in J(R)$.

Proof. (1) \Rightarrow (4) is obvious.

(2) \Leftrightarrow (3) The equation $t^2 - (u + w)t + (uw - sv) = 0$ is solvable in R if and only if the equation $(tu^{-1})^2 - (1 + wu^{-1})tu^{-1} + (wu^{-1} - svu^{-2}) = 0$ is solvable in R .

(3) \Leftrightarrow (4) Follows from Theorem 2.10.

(2) \Rightarrow (1) Let $A \in K_s(R)$. According to Theorem 2.11, we may assume that $\det_s(A) \in J(R)$ and $\text{tr}(A) \in U(R)$, and then by Lemma 2.8, similarly we can assume that $A \sim \begin{bmatrix} u & 1 \\ v & w \end{bmatrix}$ where $u, v \in U(R)$ and $w \in J(R)$. Hence, by (2) and Theorem 2.10, A is quasipolar and so holds (1). \square

Corollary 2.15. *If $R = \widehat{\mathbb{Z}}_p$, then $K_s(R)$ is quasipolar for all $s \in R$.*

Proof. We know that if $s \in U(R)$, then $K_s(R)$ is quasipolar. Let $A \in K_s(R)$ and $s \in J(R)$. In view of Theorem 2.14, we can assume that $A = \begin{bmatrix} 1 & 1 \\ u & w \end{bmatrix}$ where $u \in U(R)$ and $w \in J(R)$. Since $K_1(R)$ is strongly clean, the equation $t^2 - t - w = 0$ is solvable in R for all $w \in J(R)$ by [9, Corollary 16]. Then the equation $t^2 - t + \det_s(A)(\text{tr}(A))^{-2} = 0$ is solvable in R . This gives that the equation $t^2 - \text{tr}(A)t + \det_s(A) = 0$ is solvable in R . Hence, by Theorem 2.10, A is quasipolar in $K_s(R)$. Therefore $K_s(R)$ is quasipolar by Theorem 2.14. \square

Lemma 2.16. *Let R be a commutative local ring with $s = \sum_{i=0}^{\infty} s_i x^i \in R[[x]]$ and let $A(x) \in K_s(R[[x]])$. The following are equivalent.*

- (1) *The equation $t^2 - \text{tr}(A(0))t + \det_{s_0}(A(0)) = 0$ has a root in $J(R)$ and a root in $U(R)$.*
- (2) *The equation $t^2 - \text{tr}(A(x))t + \det_s(A(x)) = 0$ has a root in $J(R[[x]])$ and a root in $U(R[[x]])$.*

Proof. (1) \Rightarrow (2) Note that $J(R[[x]]) = J(R) + xR[[x]]$. Assume that the equation $t^2 - \text{tr}(A(0))t + \det_{s_0}(A(0)) = 0$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$. Let $y = \sum_{i=0}^{\infty} b_i x^i$, $\text{tr}(A(x)) = \sum_{i=0}^{\infty} \mu_i x^i$ and $\det_s(A(x)) = \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ where $\mu_0 = \text{tr}(A(0))$ and $\lambda_0 = \det_{s_0}(A(0))$. Then, $y^2 - \text{tr}(A(x))y - \det_s(A(x)) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$b_0^2 - b_0\mu_0 + \lambda_0 = 0 \quad (i_0)$$

$$b_1[b_0 + b_0 - \mu_0] - b_0\mu_1 + \lambda_1 = 0 \quad (i_1)$$

$$b_2[b_0 + b_0 - \mu_0] + b_1^2 - b_0\mu_2 - b_1\mu_1 + \lambda_2 = 0 \quad (i_2)$$

$$\vdots \quad \quad \quad \vdots$$

Obviously, $\mu_0 = \text{tr}(A(0)) = \alpha + \beta \in U(R)$. Let $b_0 = \alpha$. Since $b_0 + b_0 - \mu_0 = \alpha - \beta \in U(R)$, we obtain b_1, b_2, b_3, \dots from equations above. Then $t^2 - \text{tr}(A(x))t + \det_s(A(x)) = 0$ has a root $\alpha(x) \in J(R[[x]])$. If $b_0 = \beta$, analogously, we show that $t^2 - \text{tr}(A(x))t + \det_s(A(x)) = 0$ has a root $\beta(x) \in U(R[[x]])$.

(2) \Rightarrow (1) Suppose that the equation $t^2 - \text{tr}(A(x))t + \det_s(A(x)) = 0$ has a root $\alpha(x) \in J(R[[x]])$ and a root $\beta(x) \in U(R[[x]])$. This implies that the equation $t^2 - \text{tr}(A(0))t + \det_{s_0}(A(0)) = 0$ has a root $\alpha(0) \in J(R)$ and a root $\beta(0) \in U(R)$. \square

Theorem 2.17. *Let R be a commutative local ring with $s = \sum_{i=0}^{\infty} s_i x^i \in R[[x]]$. The following are equivalent.*

- (1) *$K_s(R[[x]])$ is quasipolar.*
- (2) *$K_{s_0}(R)$ is quasipolar.*

Proof. (1) \Rightarrow (2) Let $A \in K_{s_0}(R)$ with $\det_{s_0}(A) \in J(R)$. This gives $A \in K_s(R[[x]])$ and $\det_s(A) \in J(R[[x]])$. By (1) and Theorem 2.11, either $\text{tr}(A) \in J(R[[x]])$ or the equation $t^2 - \text{tr}(A)t + \det_s(A) = 0$ is solvable in $R[[x]]$. If $\text{tr}(A) \in J(R[[x]])$, then we have $\text{tr}(A) \in J(R)$ and so $A \in K_{s_0}(R)$ is quasipolar by Theorem 2.11. If the equation $t^2 - \text{tr}(A)t + \det_s(A) = 0$ is solvable in $R[[x]]$, then the equation $t^2 - \text{tr}(A)t + \det_{s_0}(A) = 0$ is solvable in R by Lemma 2.16. So $A \in K_{s_0}(R)$ is quasipolar by Theorem 2.11.

(2) \Rightarrow (1) Let $A(x) \in K_s(R[[x]])$ with $\det_s(A(x)) \in J(R[[x]])$. This gives $\det_{s_0}(A(0)) \in J(R)$. If $\text{tr}(A(x)) \in J(R[[x]])$, then $A(x)$ is quasipolar by Theorem 2.10. Hence we can assume that $\text{tr}(A(x)) \in U(R[[x]])$ and so $\text{tr}(A(0)) \in U(R)$. By (2) and Lemma 2.16, the equation $t^2 - \text{tr}(A(0))t + \det_{s_0}(A(0)) = 0$ is solvable in R . Let $\lambda, \mu \in R$ be roots of this equation. Since $\text{tr}(A(0)) \in U(R)$ and $\det_{s_0}(A(0)) \in J(R)$, one of λ, μ must be in $U(R)$ and the other must be in $J(R)$. Without loss of generality we assume that $\lambda \in J(R)$ and $\mu \in U(R)$. Thus the equation $t^2 - \text{tr}(A(x))t + \det_s(A(x)) = 0$ is solvable in $R[[x]]$ by Lemma 2.16. According to Theorem 2.11, $A(x) \in K_s(R[[x]])$ is quasipolar and so holds (1). \square

Corollary 2.18. *Let R be a commutative local ring with $s \in R$. Then $K_s(R[[x]])$ is quasipolar if and only if $K_s(R)$ is quasipolar.*

Corollary 2.19. *Let R be a commutative local ring. Then the following holds.*

- (1) $K_0(R)$ is quasipolar.
- (2) $K_s(R[[x]])$ is quasipolar for all $s \in xR[[x]]$.

Proof. (1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(R)$ where $a, b, c, d \in R$. By Theorem 2.11, we can assume that $\det_0(A) = ad \in J(R)$ and $\text{tr}(A) = a + d \in U(R)$. This implies that $a - d \in U(R)$. Choose $P = \begin{bmatrix} 1 & 0 \\ c(a-d)^{-1} & -1 \end{bmatrix}$. Hence $P \in U(K_0(R))$ by Lemma 2.2(2) and a simple computation shows that $P^{-1}AP = \begin{bmatrix} a & -b \\ 0 & d \end{bmatrix}$. By Lemma 2.4 and Lemma 2.9, A is quasipolar. (2) is obvious from (1) and Theorem 2.17. \square

Note that $M_2(\mathbb{Z}_{(2)})$ is not quasipolar but $K_0(\mathbb{Z}_{(2)})$ is quasipolar by Corollary 2.19.

Theorem 2.20. *Let R be a commutative local ring with $s = \sum_{i=0}^{n-1} s_i x^i \in R[x]/(x^n)$, where $n \geq 1$. The following are equivalent.*

- (1) $K_s(R[x]/(x^n))$ is quasipolar.
- (2) $K_{s_0}(R)$ is quasipolar.

In particular, for all $1 \leq m \leq n$, $K_{x^m}(R[x]/(x^n))$ is quasipolar.

Proof. Similar to the proof of Theorem 2.17. □

Theorem 2.21. *Let R be a commutative local ring. If $s \in R$ is nilpotent, then $K_s(R)$ is quasipolar.*

Proof. Let $A \in K_s(R)$. We may assume that $A \notin U(K_s(R))$ and $A \notin (K_s(R))^{qnil}$. According to Lemma 2.8, A is similar to $\begin{bmatrix} u & 1 \\ v & w \end{bmatrix}$ or $\begin{bmatrix} w & 1 \\ v & u \end{bmatrix}$ where $u, v \in U(R)$ and $w \in J(R)$. Without loss of generality, $A \sim B = \begin{bmatrix} u & 1 \\ v & w \end{bmatrix}$. Note that $t^2 - \text{tr}(B)t + \det_s(B) = t^2 - (u + w)t + (uw - sv) = 0$ if and only if $((u + w)t)^2 - (u + w)((u + w)t) + uw - sv = (u + w)^2[t^2 - t + (uw - sv)(u + w)^{-2}] = 0$. In proof of [9, Theorem 22], it is proved that the equation $t^2 - ut - w = 0$ is solvable in R for all $u \in 1 + J(R)$ and $w \in J(R)$. Then the equation $t^2 - t + (uw - sv)(u + w)^{-2} = 0$ is solvable in R and so the equation $t^2 - \text{tr}(B)t + \det_s(B) = 0$ is solvable in R . In view of Theorem 2.10, A is quasipolar and so $K_s(R)$ is quasipolar. □

For a commutative local ring R , $K_s(R)$ is quasipolar for some non-nilpotent elements s in $J(R)$, by Corollary 2.19.

Example 2.22. If $R = \mathbb{Z}_{(p)}[x]/(x^2)$, then $K_s(R)$ is quasipolar if and only if $s \in R^{nil}$.

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